

Lower and upper bounds of martingale measure densities in continuous time markets

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Abstract

In a continuous time market model we consider the problem of existence of an equivalent martingale measure with density lying within given lower and upper bounds and we characterize a necessary and sufficient condition for this. In this sense our main result can be regarded as a version of the fundamental theorem of asset pricing. In our approach we suggest an axiomatic description of prices on L_p -spaces (with $p \in [1, \infty)$) and we rely on extension theorems for operators.

Key-words: equivalent martingale measures, fundamental theorem, extension theorem, asset pricing.

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1 Introduction

Two key concepts in mathematical finance are those of *arbitrage* and *equivalent martingale measures*. An arbitrage is an opportunity to obtain a riskless profit with positive probability. It is clear that such opportunities cannot prevail in a market where all the agents have the same information and trading possibilities. An equivalent martingale measure is a probability measure under which the current discounted asset price is the expected future discounted payoff of the asset. This measure is equivalent to the "real world" physical measure in the sense that they assign zero probability to the same events.

The connection between the existence of an equivalent martingale measure and the absence of arbitrage is the subject of the various versions of *the fundamental theorem of asset pricing* - see for example the seminal papers [6], [15] and the recent survey monograph [7]. This theorem states that for a

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fairly wide range of price processes, the existence of an equivalent martingale measure rules out not only arbitrage opportunities but also the possibility of obtaining a "free lunch with vanishing risk" - a condition that in turn ensures the existence of an equivalent martingale measure.

We have to stress that such a measure is not necessarily unique, it is shown in [13] that (provided existence) *the equivalent martingale measure is unique if and only if the market is complete* - that is, if every claim is attainable. As a consequence the no-arbitrage principle does not give a unique price in an incomplete market, but a whole range of prices that are equally valid from the no-arbitrage point of view.

Thus many authors have been engaged in finding properties of equivalent martingale measures that either make them in some sense optimal or justify their use in specific incomplete market models. Without aim or possibility to be complete we mention the *minimal martingale measure* and *variance-optimal martingale measure* (see [20]) which are both in some sense minimizing the distance to the physical measure. In [12] utility arguments are used to justify the so-called *Esscher measure*. In [5] the structure-preserving properties are emphasized.

Instead of searching for the unique "optimal" equivalent martingale measure, one can try to characterize probability measures that are in some sense "reasonable". In [4] (see also [3], [21]) bounds on the Sharpe ratio (the ratio of the risk premium to the volatility) are used to restrict the set of equivalent martingale measures such that they rule out not only arbitrage opportunities but also deals that are "too good".

In some applications, one deals with events of crucial nature occurring with small but non-zero probability. See for example the pricing of (re)insurance linked products. In this sense "reasonable" measures should "preserve small probabilities", i.e.

$$"P(A) \text{ small}" \Leftrightarrow "P^0(A) \text{ small}."$$

This appears to be of priority importance, in fact the assessment under P of the risk of these events incurring can be seriously misjudged under a P^0 only equivalent to P . With this motivation in mind we study the characterisation of the existence of equivalent probability measures P^0 with densities $\frac{dP^0}{dP}$ lying within *pre-considered* lower and upper bounds:

$$0 < m \leq \frac{dP^0}{dP} \leq M < \infty \quad P\text{-a.s.} \quad (1.1)$$

We stress that the above bounds m, M are *random variables*.

In [19] a study on the existence of a martingale measure with lower bounded density is traced. Lower bounds for martingale measure densities are also considered in [18]. In [14] densities are bounded from above. However, the goal in this study is to show that the set of equivalent σ -martingale measures with density in $L_\infty(\mathcal{F})$ is dense (in total variation) in the set of equivalent σ -martingale measures. See also [17].

In the present paper we consider lower and upper bounds for martingale measure densities simultaneously. Our main result (see Theorem 4.1) gives a necessary and sufficient condition for its existence and in this sense it is a version of the fundamental theorem of asset pricing.

Our approach relies on a bounds preserving extension theorem for operators first proved in [1]. A first version of the fundamental theorem of asset pricing with lower and upper bounds for the density for a single-period market model was also given in [1], while the multi-period case was just traced. See also [8].

Our paper extends these results to the continuous-time model. We remark that to perform this extension we have introduced an axiomatic approach to the definition of price processes and of a "time-consistent" family of price processes. See Definition 2.1 and Definition 2.4. This characterisation allows for a model independent treatment of prices. The approach taken is inspired by the axiomatic approach to risk measures, see e.g. [2], [10].

The paper is organized as follows. In Section 2 we introduce the claims and price operators. The no-arbitrage pricing and some useful representation and extension results for price operators are presented in Section 3. Section 4 is dedicated to our main result. Some examples are provided in Section 5.

2 Framework, claims and price operators

We consider a continuous time market model without frictions on the time interval $[0, T]$, $T > 0$. Let (Ω, \mathcal{F}, P) be a complete probability space equipped with the *right-continuous* filtration $\mathbb{F} := \{\mathcal{F}_t, 0 \leq t \leq T\}$ with $\mathcal{F}_T = \mathcal{F}$. Unless otherwise noted, expectations, almost surely statements etc. are with respect to the *physical measure* P .

We will work in an L_p -framework, and consider claims as elements of the separable space $L_p(\mathcal{F}_t) := L_p(\Omega, \mathcal{F}_t, P)$ with finite norm

$$\|X\|_p := E[|X|^p]^{1/p}, \quad X \in L_p(\mathcal{F}_t),$$

for some $1 \leq p < \infty$. We will use the superscript $^+$ to denote sets of non-

negative elements. The L_p -spaces are considered as *complete lattices* (see e.g. [22]) where " \geq " is the standard point-wise relation " \geq P -a.s.". In this framework we also consider the strict relation " $>$ " which means that, in addition to " \geq P -a.s.", the point-wise relation " $>$ " holds on a set $A \in \mathcal{F}$ such that $P(A) > 0$. This choice allows for a more flexible comparison between claims, see e.g. the forthcoming (2.2b).

To be able to compare prices over time we consider a *numéraire* R_t , $t \in [0, T]$, representing the "unit of measurement" of money. This is an asset which is always available at price $R_t > 0$ P -a.s. for every t . To simplify notation we assume that $R_t \equiv 1$, $t \in [0, T]$, and to ease the terminology we use the terms price operators and prices even when we consider *discounted* prices.

2.1 Market claims

For any time $t \in [0, T]$, let

$$L_t^+ \subseteq L_p^+(\mathcal{F}_t) \quad (2.1)$$

denote the *convex sub-cone* representing all *market* claims that are payable at time t ($0 \in L_t^+$). Note that in a *complete* market $L_t^+ = L_p(\mathcal{F}_t)$ for all $t \in [0, T]$. However, in general $L_t^+ \subsetneq L_p^+(\mathcal{F}_t)$ for some $t \in [0, T]$.

2.2 Price operators

We refer to a claim $X \in L_t^+$ as *available* at time $s \leq t$, if it can be bought at time s at the \mathcal{F}_s -measurable price $x_{st}(X) < \infty$ P -a.s.

Definition 2.1. *For any fixed $s, t \in [0, T]$, $s \leq t$, the operator $x_{st}(X)$, $X \in L_t^+$, is a price operator if it is*

- strictly monotone, i.e. for any $X', X'' \in L_t^+$ available at s

$$x_{st}(X') \geq x_{st}(X''), \quad X' \geq X'', \quad (2.2a)$$

$$x_{st}(X') > x_{st}(X''), \quad X' > X'', \quad (2.2b)$$

- additive, i.e. for any $X', X'' \in L_t^+$ available at s

$$x_{st}(X' + X'') = x_{st}(X') + x_{st}(X'') \quad X', X'' \in L_t^+, \quad (2.3)$$

- and \mathcal{F}_s -homogeneous, i.e.

$$x_{st}(\lambda X) = \lambda x_{st}(X) \quad (2.4)$$

for all $X \in L_t^+$ available at s and \mathcal{F}_s -measurable multipliers λ such that $\lambda X \in L_t^+$.

From (2.3), we have that $x_{st}(0) = 0$. Moreover, note that the requirement of additivity (2.3) is also necessary in view of the no-arbitrage argument that will follow. Naturally, we set $x_{tt}(X) = X$, $X \in L_t^+$. Since the numéraire is always available, $1 \in L_t^+$ and $x_{st}(1) = 1$.

Remark 2.1. *This axiomatic approach to price processes is inspired by risk measure theory. The requirements (2.2a),(2.3),(2.4) are related to coherent risk measures. The additional assumption of strict monotonicity (2.2b), is related to relevant risk measures. See e.g. [2], [10].*

Remark 2.2. *Note that there is a unique extension of x_{st} as a price operator to the subspace*

$$L_t := L_t^+ - L_t^+$$

of the elements of $L_p(\mathcal{F}_t)$ which can be expressed as

$$X = X' - X''$$

for some $X', X'' \in L_t^+$. This extension is given by

$$x_{st}(X) := x_{st}(X') - x_{st}(X'').$$

In financial terms this corresponds to allowing short selling of the market claims.

Definition 2.2. *The price operator $x_{st}(X)$, $X \in L_t^+$, is tame if*

$$x_{st}(X) \in L_p^+(\mathcal{F}_s), \quad X \in L_t^+, \quad (2.5)$$

i.e. $\|x_{st}(X)\|_p < \infty$, $X \in L_t^+$.

Let us consider the family of price operators of $X \in L_t^+$, $t \leq T$,

$$x_{st}(X), \quad 0 \leq s \leq t. \quad (2.6)$$

Definition 2.3. *The family (2.6) is right-continuous at s if X is available for some interval of time $[s, s + \delta]$ ($\delta > 0$) and*

$$\|x_{s't}(X) - x_{st}(X)\|_p \rightarrow 0, \quad s' \downarrow s. \quad (2.7)$$

Definition 2.4. *Let $\mathcal{T} \subseteq [0, T]$. The family x_{st} , $s, t \in \mathcal{T} : s \leq t$, of tame discounted price operators $x_{st}(X)$, $X \in L_t^+$, is time-consistent (in \mathcal{T}) if for all $s, u, t \in \mathcal{T} : s \leq u \leq t$*

$$x_{st}(X) = x_{su}(x_{ut}(X)), \quad (2.8)$$

for all $X \in L_t^+$ such that $x_{ut}(X) \in L_u^+$.

In the sequel we will consider time-consistency (2.8). This is a natural assumption in view of standard arguments of absence of arbitrage.

3 No-arbitrage pricing, representation and extension theorems

Financial pricing rules are governed by the principle of *no arbitrage* ruling out the possibility of earning a riskless profit. The absence of arbitrage is ensured by the existence of an *equivalent risk neutral probability measure*, $P^0 \sim P$, such that the *prices* $x_{st}(X)$, $X \in L_t^+$, admit the representation

$$x_{st}(X) = E^0[X|\mathcal{F}_s], \quad X \in L_t^+. \quad (3.1)$$

For any $t \in [0, T]$ and $X \in L_t^+$ the *price process*

$$x_{st}(X), \quad 0 \leq s \leq t,$$

is a *martingale* with respect to the measure P^0 and the filtration \mathbb{F} . For this reason measures under which (3.1) holds are referred to as *equivalent martingale measures*.

Definition 3.1. A probability measure $P^0 \sim P$ is tame if for all $t \in [0, T]$,

$$E^0[X|\mathcal{F}_t] \in L_p(\mathcal{F}_t), \quad X \in L_p(\mathcal{F}).$$

If $P^0 \sim P$ is a tame probability measure, then the conditional expectation

$$E^0[\cdot|\mathcal{F}_s] : L_p(\mathcal{F}_t) \longrightarrow L_p(\mathcal{F}_s)$$

is a tame strictly monotone, linear, \mathcal{F}_s -homogeneous operator and hence it has all the properties of a tame price operator on the whole $L_p^+(\mathcal{F}_s)$ (and $L_p(\mathcal{F}_t)$ by Remark 2.2). Clearly the family of conditional expectations satisfies (2.8):

$$E^0[X|\mathcal{F}_s] = E^0[E^0[X|\mathcal{F}_u]|\mathcal{F}_s], \quad X \in L_p(\mathcal{F}_t), \quad 0 \leq s \leq u \leq t,$$

and thanks to the right-continuity of the filtration also (2.7) holds.

Quite remarkably, the converse is also true: all the tame price operators $x_{su}(X)$, $X \in L_p(\mathcal{F}_u)$, with $0 \leq s \leq u \leq t$, admit representation as conditional expectation with respect to the same equivalent martingale measure. See Theorem 3.1.

3.1 Representation theorems

The following lemma summarizes results first proved in [1] and [9]. To keep the exposition self-contained we briefly sketch the proof.

Lemma 3.1. Fix $s, t \in [0, T]$: $s \leq t$. The operator $x_{st}(X)$, $X \in L_p(\mathcal{F}_t)$, is tame, strictly monotone, linear, and \mathcal{F}_s -homogeneous if and only if it admits representation

$$x_{st}(X) = E_{st}^0[X|\mathcal{F}_s], \quad X \in L_p(\mathcal{F}_t), \quad (3.2)$$

with respect to a tame probability measure

$$P_{st}^0(A) = \int_A f_{st}(\omega) P(d\omega), \quad A \in \mathcal{F}_t, \quad (3.3)$$

where $f_{st} \in L_q^+(\mathcal{F}_t)$, $\frac{1}{q} + \frac{1}{p} = 1$ with $f_{st} > 0$ P -a.s. In addition, the operator (3.2) is bounded (continuous) if and only if

$$\begin{cases} \operatorname{esssup} E \left[\left(\frac{f_{st}}{E[f_{st}|\mathcal{F}_s]} \right)^q \middle| \mathcal{F}_s \right] < \infty, & p \in (1, \infty) \\ \operatorname{esssup} \frac{f_{st}}{E[f_{st}|\mathcal{F}_s]} < \infty, & p = 1. \end{cases} \quad (3.4)$$

Proof. Recall that $x_{st}(X)$, $X \in L_p(\mathcal{F}_t)$, being defined on the whole space, is continuous (cf. [11]). Define

$$\phi(X) := E[x_{st}(X)], \quad X \in L_p(\mathcal{F}_t). \quad (3.5)$$

This is a linear, strictly monotone, and continuous functional. By the Riesz representation theorem, there exists a unique element $f_{st} \in L_q(\mathcal{F}_t)$, $\frac{1}{q} + \frac{1}{p} = 1$, such that

$$\phi(X) = E[X f_{st}], \quad X \in L_p(\mathcal{F}_t). \quad (3.6)$$

The strict monotonicity ensures that $f \in L_q^+(\mathcal{F}_t)$ and $f_{st} > 0$ P -a.s. Moreover, $E[f_{st}] = \phi(1) = E[x_{st}(1)] = 1$. Since x_{st} is \mathcal{F}_s -homogeneous, we have

$$\phi(\chi_B x_{st}(X)) = E[x_{st}(x_{st}(\chi_B X))] = E[x_{st}(\chi_B X)] = \phi(\chi_B X), \quad B \in \mathcal{F}_s.$$

Namely, $E[\chi_B x_{st}(X) f_{st}] = E[\chi_B X f_{st}]$. Then, we have $E[\chi_B x_{st}(X) E[f_{st}|\mathcal{F}_s]] = E[\chi_B E[X f_{st}|\mathcal{F}_s]]$ and

$$x_{st}(X) = E \left[X \frac{f_{st}}{E[f_{st}|\mathcal{F}_s]} \middle| \mathcal{F}_s \right] = E_{st}^0[X|\mathcal{F}_s]. \quad (3.7)$$

Representation (3.7) shows that P_{st}^0 is tame as x_{st} is tame. Hence $x_{st}(X)$, $X \in L_p(\mathcal{F}_t)$, admits the representation (3.2) with respect to the measure (3.3). The converse is true.

The Hölder equality for conditional expectations (see [9] and [1, Theorem 2.1]) provides the evaluation of the norm $|||x_{st}|||$ for the operator (3.7):

$$|||x_{st}||| := \sup_{\|X\|_p \leq 1} \|x_{st}(X)\|_p = \begin{cases} \operatorname{esssup} E \left[\left(\frac{f_{st}}{E[f_{st}|\mathcal{F}_s]} \right)^q \middle| \mathcal{F}_s \right]^{1/q}, & p \in (1, \infty) \\ \operatorname{esssup} \frac{f_{st}}{E[f_{st}|\mathcal{F}_s]}, & p = 1. \end{cases}$$

Thus x_{st} is bounded if and only if (3.4) is satisfied. \square

We remark that the representation (3.7) is not unique. In fact the following result holds.

Lemma 3.2. *Fix $s, t \in [0, T]$: $s \leq t$. The tame operator $x_{st}(X)$, $X \in L_p(\mathcal{F}_t)$, is strictly monotone linear \mathcal{F}_s -homogeneous if and only if it admits representation*

$$x_{st}(X) = \tilde{E}_{st}^0[X|\mathcal{F}_s], \quad X \in L_p(\mathcal{F}_t), \quad (3.8)$$

with respect to a tame probability measure

$$\tilde{P}_{st}^0(A) := E[x_{st}(\chi_A)f_{st}], \quad A \in \mathcal{F}_t, \quad (3.9)$$

where f_{st} is the density in (3.3). Moreover, $\tilde{P}_{st}^0 \sim P$ and the density $h_{st} = \frac{d\tilde{P}_{st}^0}{dP} \in L_q^+(\mathcal{F}_t)$ satisfies

$$E[h_{st}|\mathcal{F}_s] = E[f_{st}|\mathcal{F}_s]. \quad (3.10)$$

Proof. By the Riesz representation theorem applied to the strictly monotone continuous functional

$$\psi(X) = E[x_{st}(X)f_{st}] = E[x_{st}(X)E[f_{st}|\mathcal{F}_s]], \quad X \in L_p(\mathcal{F}_t),$$

there exists $h_{st} \in L_q(\mathcal{F}_t)$ ($\frac{1}{q} + \frac{1}{p} = 1$) with $h_{st} > 0$ P -a.s. such that $\psi(X) = E[Xh_{st}]$. Following similar arguments as in the proof of Lemma 3.1 we can see that

$$x_{st}(X) = E\left[X \frac{h_{st}}{E[f_{st}|\mathcal{F}_s]} \middle| \mathcal{F}_s\right], \quad X \in L_p(\mathcal{F}_t). \quad (3.11)$$

Taking $X = 1$ we see that $E[h_{st}|\mathcal{F}_s] = E[f_{st}|\mathcal{F}_s]$. The probability measure (3.9) admits the following equivalent representations:

$$\tilde{P}_{st}^0(A) = E[x_{st}(\chi_A)f_{st}] = E[x_{st}(\chi_A)h_{st}] = E[\chi_A h_{st}], \quad A \in \mathcal{F}_t.$$

Thus $\tilde{P}_{st}^0 \sim P$ and $\tilde{P}_{st}^0(A) = \int_A h_{st} P(d\omega)$, $A \in \mathcal{F}_t$. The converse is also true. \square

Remark 3.1. *In view of Lemma 3.2 we see that there is a unique representation if and only if $E[f_{st}|\mathcal{F}_s] = 1$.*

The lemmas above consider two fixed time points $s, t \in [0, T]$: $s \leq t$. In the following result we consider $s \in [0, T]$ fixed and we compare the representations of x_{su} , x_{st} for $s \leq u \leq t$.

Theorem 3.1. *Assume that the operators*

$$x_{su}(X), \quad X \in L_p(\mathcal{F}_u), \quad s \leq u \leq t, \quad (3.12)$$

are tame price operators constituting a time-consistent family. Then, for all $u \in [s, t]$, the representation

$$x_{su}(X) = E_{st}^0[X|\mathcal{F}_s], \quad X \in L_p(\mathcal{F}_u), \quad (3.13)$$

holds in terms of the tame measure P_{st}^0 defined on (Ω, \mathcal{F}_t) , cf. (3.3). Moreover $P_{st}^0|_{\mathcal{F}_u} = P_{su}^0$, for all $u \in [s, t]$.

Proof. Let us consider $u \in [s, t]$ and $X \in L_p(\mathcal{F}_u) \subseteq L_p(\mathcal{F}_t)$. By time-consistency, the \mathcal{F}_u -homogeneity of x_{ut} , and Lemma 3.1 we have

$$x_{su}(X) = x_{su}(x_{ut}(X)) = x_{st}(X) = E\left[X \frac{f_{st}}{E[f_{st}|\mathcal{F}_s]} \middle| \mathcal{F}_s\right] = E_{st}^0[X|\mathcal{F}_s]$$

which proves (3.13). Furthermore, from (3.5)-(3.6) we have

$$E[Xf_{st}] = E[x_{st}(X)] = E[x_{su}(X)] = E[Xf_{su}]$$

for all $X \in L_p(\mathcal{F}_u)$. Thus $E[f_{st}|\mathcal{F}_u] = f_{su}$. Hence, for any $A \in \mathcal{F}_u$, we have $P_{st}^0(A) = E[\chi_A f_{st}] = E[\chi_A f_{su}] = P_{su}^0(A)$. Namely, $P_{st}^0|_{\mathcal{F}_u} = P_{su}^0$. \square

Corollary 3.1. *The representation (3.13) is equivalent to*

$$x_{su}(X) = \tilde{E}_{st}^0[X|\mathcal{F}_s], \quad X \in L_p(\mathcal{F}_u), \quad (3.14)$$

by means of the measure (3.9) and $\tilde{P}_{st}^0|_{\mathcal{F}_u} = \tilde{P}_{su}^0$.

Proof. By application of Lemma 3.2 and Theorem 3.1 we have:

$$\begin{aligned} \tilde{P}_{st}^0(A) &= E[x_{st}(\chi_A)f_{st}] = E[x_{su}(\chi_A)E[f_{st}|\mathcal{F}_u]] \\ &= E[x_{su}(\chi_A)f_{su}] = \tilde{P}_{su}^0(A). \end{aligned}$$

\square

As seen, whenever we have a time-consistent family of tame price operators $x_{st}(X)$, $0 \leq s \leq t \leq T$, defined on the *whole* cone $X \in L_p^+(\mathcal{F}_t)$, we have an equivalent martingale measure. This is always the case in markets that are complete. However, in general, operators are defined on the sub-cones $L_t^+ \subseteq L_p^+(\mathcal{F}_t)$. Then the existence of an equivalent martingale measure is linked to the admissibility of an extension of the price operator from the sub-cones to the corresponding cones.

3.2 Extension theorems

In [1] some extension theorems for operators are presented. These theorems may be regarded as versions of the Hahn-Banach extension theorems for linear operators. Actually the results concern operators in L_p -spaces which are bounded simultaneously from above and from below and the extension is bounds preserving. In this sense they can be regarded as versions of the König theorem (see e.g. [11]). We now review briefly these results in a form that suits the arguments to come.

Let \mathcal{A} and \mathcal{B} be σ -algebras such that $\mathcal{B} \subseteq \mathcal{A}$ and consider a general monotone linear operator

$$x : L^+ \longrightarrow L_p^+(\mathcal{B}) \quad (3.15)$$

defined on the convex sub-cone

$$L^+ \subseteq L_p^+(\mathcal{A}).$$

We will exploit the ordering and the lattice structure mentioned in Section 2. Consider the monotone operators

$$m, M : L_p^+(\mathcal{A}) \longrightarrow L_p^+(\mathcal{B})$$

such that

$$m(X) \leq x(X) \leq M(X), \quad X \in L^+, \quad (3.16)$$

where the *minorant* m is super-linear, i.e.

$$\begin{aligned} m(\lambda X) &= \lambda m(X), & X \in L_p^+(\mathcal{A}), \lambda \geq 0 \\ m(X' + X'') &\geq m(X') + m(X''), & X', X'' \in L_p^+(\mathcal{A}) \end{aligned}$$

and the *majorant* M is sub-linear, i.e.

$$\begin{aligned} M(\lambda X) &= \lambda M(X), & X \in L_p^+(\mathcal{A}), \lambda \geq 0 \\ M(X' + X'') &\leq M(X') + M(X''), & X', X'' \in L_p^+(\mathcal{A}). \end{aligned}$$

We say that the operator satisfies a *sandwich condition* if

$$m(Y'') + x(X'') \leq x(X') + M(Y') \quad (3.17)$$

for all $X', X'' \in L^+, Y', Y'' \in L_p^+(\mathcal{A})$: $Y'' + X'' \leq X' + Y'$.

In particular, if $L^+ = L_p^+(\mathcal{A})$, then (3.17) is equivalent to (3.16). This gives the justification for the term "sandwich condition".

Theorem 3.2. ([1, Theorem 3.2, Theorem 5.1, Corollary 5.1, Theorem 5.2]) *The monotone (2.2a), additive, operator x in (3.15) satisfying (3.17) with respect to some super-linear minorant m and sub-linear majorant M admits a monotone, additive extension*

$$x : L_p^+(\mathcal{A}) \longrightarrow L_p^+(\mathcal{B}). \quad (3.18)$$

The extension is sandwich preserving, which means that (3.16) holds on the entire $L_p^+(\mathcal{A})$.

Moreover the operator (3.15) and its extension (3.18) are strictly monotone if and only if (3.17) holds for some strictly positive minorant, i.e.

$$X > 0 \implies m(X) > 0$$

in the given sense (Section 2). If the majorant M is \mathcal{B} -homogeneous, then the operator (3.18) is \mathcal{B} -homogeneous.

Remark 3.2. *If the monotone, additive, operator (3.15)-(3.16) admits the monotone, additive sandwich preserving extension (3.18), then certainly (3.17) holds.*

Now we return to the tame price operators introduced in Section 2.2. Fix some $s, t \in [0, T]$, $s \leq t$, and consider *linear* minorants and majorants (3.16) of the form

$$\begin{aligned} m_{st}(X) &= E[Xm_{st}|\mathcal{F}_s], & X &\in L_p(\mathcal{F}_t), \\ M_{st}(X) &= E[XM_{st}|\mathcal{F}_s], & X &\in L_p(\mathcal{F}_t), \end{aligned} \quad (3.19)$$

where the random variables $m_{st}, M_{st} \in L_q(\mathcal{F}_t)$, $\frac{1}{p} + \frac{1}{q} = 1$ are such that $0 < m_{st} \leq M_{st}$ P -a.s. The sandwich condition (3.17) is now written

$$E[Y''m_{st}|\mathcal{F}_s] + x_{st}(X'') \leq x_{st}(X') + E[Y'M_{st}|\mathcal{F}_s] \quad (3.20)$$

for all $X', X'' \in L_t^+$, $Y', Y'' \in L_p^+(\mathcal{F}_t)$: $Y'' + X'' \leq X' + Y'$.

As a consequence of Lemma 3.1 and Theorem 3.2, we have

Theorem 3.3. *([1, Theorem 5.2, Corollary 5.2], [8, Theorem 2]) The tame price operator*

$$x_{st} : L_t^+ \longrightarrow L_p^+(\mathcal{F}_s) \quad (3.21)$$

satisfying (3.20) admits a tame strictly monotone, additive, \mathcal{F}_s -homogeneous, sandwich preserving extension

$$x_{st} : L_p^+(\mathcal{F}_t) \longrightarrow L_p^+(\mathcal{F}_s). \quad (3.22)$$

Moreover the extension admits the representation (3.2)-(3.3):

$$x_{st}(X) = E_{st}^0[X|\mathcal{F}_s]$$

with respect to the tame probability measure $P_{st}^0 \sim P$ with density $\frac{dP_{st}^0}{dP} = f_{st}$. Furthermore,

$$m_{st} \leq \frac{f_{st}}{E[f_{st}|\mathcal{F}_s]} \leq M_{st}. \quad (3.23)$$

If

$$\begin{cases} \text{esssup } E[M_{st}^q|\mathcal{F}_s] < \infty, & p > 1, \\ \text{esssup } M_{st} < \infty, & p = 1. \end{cases} \quad (3.24)$$

holds, then the operator (3.22) is bounded (continuous).

Remark 3.3. *The converse holds true. The existence of an extension (3.22) for (3.21) which then has a representation of the form (3.2)-(3.3) where f_{st} satisfies (3.23) ensures that the price operator (3.22) satisfies (3.16) on $L_p^+(\mathcal{F}_t)$.*

4 A version of the fundamental theorem of asset pricing

Let $m, M \in L_q^+(\mathcal{F})$ such that $0 < m \leq M$ P -a.s. For any $0 \leq s \leq t \leq T$ consider the elements $0 < m_{st} \leq M_{st}$ P -a.s. of $L_q^+(\mathcal{F}_t)$ such that

$$m = m_{0T} = m_{0s} \cdot m_{st} \cdot m_{tT}, \quad M = M_{0T} = M_{0s} \cdot M_{st} \cdot M_{tT}. \quad (4.1)$$

For example, if $\frac{m}{E[m|\mathcal{F}_s]} \leq \frac{M}{E[M|\mathcal{F}_s]}$ in $L_q^+(\mathcal{F})$, then we can define

$$m_{st} := \left(E[m|\mathcal{F}_0]\right)^{\frac{t-s}{T}} \frac{E[m|\mathcal{F}_t]}{E[m|\mathcal{F}_s]}$$

$$M_{st} := \left(E[M|\mathcal{F}_0]\right)^{\frac{t-s}{T}} \frac{E[M|\mathcal{F}_t]}{E[M|\mathcal{F}_s]}.$$

Theorem 4.1. *Let*

$$x_{st}(X), X \in L_t^+, 0 \leq s \leq t \leq T, \quad (4.2)$$

be a time-consistent (2.8) and right-continuous (2.7) family of tame price operators. Suppose that every $x_{st}(X)$, $X \in L_t^+$, satisfies the sandwich condition (3.20), i.e.

$$E[Y''m_{st}|\mathcal{F}_s] + x_{st}(X'') \leq x_{st}(X') + E[Y'M_{st}|\mathcal{F}_s]$$

for all $X', X'' \in L_t^+, Y', Y'' \in L_p^+(\mathcal{F}_t)$ such that $Y'' + X'' \leq X' + Y'$. Then there exists a tame probability measure $P^0 \sim P$:

$$P^0(A) = \int_A f(\omega) P(d\omega), \quad A \in \mathcal{F},$$

with $f \in L_q^+(\mathcal{F})$ such that $E[f|\mathcal{F}_0] = 1$ and

$$0 < m \leq f \leq M \quad P - a.s. \quad (4.3)$$

allowing the representation

$$x_{st}(X) = E\left[X \frac{f}{E[f|\mathcal{F}_s]}|\mathcal{F}_s\right] = E^0[X|\mathcal{F}_s], \quad X \in L_t^+,$$

for all price operators. The converse is also true.

Proof. We have to prove that the set of probability measures

$$\mathbb{P} := \left\{ P^0 \mid \frac{dP^0}{dP} = f, E[f|\mathcal{F}_0] = 1, m \leq f \leq M : \right. \\ \left. \forall s, t \in [0, T], s \leq t, x_{st}(X) = E^0[X|\mathcal{F}_s] \forall X \in L_t^+ \right\} \quad (4.4)$$

is non-empty if (3.20) holds. First of all note that by Theorem 3.3, the sandwich condition (3.20) ensures that for every $s \leq t$ the price operators (4.2) admit extensions (3.22) with the representation (3.2)-(3.3), i.e.

$$x_{st}(X) = E\left[X \frac{f_{st}}{E[f_{st}|\mathcal{F}_s]} \middle| \mathcal{F}_s\right], \quad X \in L_p(\mathcal{F}_t). \quad (4.5)$$

However, we remark that though the family of operators (4.2) is time-consistent, we cannot say, in general, that the extensions (4.5) are also time-consistent. Thus we cannot directly apply Theorem 3.1 to conclude. Instead we consider at first the discrete time case

$$\mathbb{P}^{(\mathcal{T})} := \left\{ P^0 \middle| \frac{dP^0}{dP} = f, E[f|\mathcal{F}_0] = 1, m \leq f \leq M : \right. \\ \left. \forall s \in \mathcal{T}, t \in [s, T], x_{st}(X) = E^0[X|\mathcal{F}_s] \forall X \in L_t^+ \right\},$$

where \mathcal{T} is some partition of $[0, T]$ of the form

$$\mathcal{T} = \{s_0, s_1, \dots, s_K\}, \text{ with } 0 = s_0 < s_1 < \dots < s_K = T. \quad (4.6)$$

Further, we consider a sequence $\{\mathcal{T}_n\}_{n=1}^\infty$ of increasingly refined partitions, such that $\mathcal{T}_n \subset \mathcal{T}_{n+1}$ and $\text{mesh}(\mathcal{T}_n) \rightarrow 0$ as $n \rightarrow \infty$. Clearly $\mathbb{P}^{(\mathcal{T}_{n+1})} \subset \mathbb{P}^{(\mathcal{T}_n)}$. It is then sufficient to prove that

- A. $\mathbb{P}^{(\mathcal{T})}$ is non-empty for any finite partition \mathcal{T} ,
- B. the infinite intersection $\bigcap_{n=1}^\infty \mathbb{P}^{(\mathcal{T}_n)}$ is non-empty, and
- C. any $P^0 \in \bigcap_{n=1}^\infty \mathbb{P}^{(\mathcal{T}_n)}$ is also in \mathbb{P} .

Let us consider the partition points \mathcal{T} and define

$$f := \prod_{k=1}^K \frac{f_{s_{k-1}s_k}}{E[f_{s_{k-1}s_k}|\mathcal{F}_{s_{k-1}}]}. \quad (4.7)$$

Note that by Lemma 3.2 (Remark 3.1) we can choose $E[f_{s_{k-1}s_k}|\mathcal{F}_{s_{k-1}}] = 1$, $k = 1, \dots, K$. Then

$$\begin{aligned} x_{s_{k-1}s_k}(X) &= E\left[X \frac{f_{s_{k-1}s_k}}{E[f_{s_{k-1}s_k}|\mathcal{F}_{s_{k-1}}]} \middle| \mathcal{F}_{s_{k-1}}\right] \\ &= E\left[X \frac{f}{E[f|\mathcal{F}_{s_{k-1}}]} \middle| \mathcal{F}_{s_{k-1}}\right], \quad X \in L_p(\mathcal{F}_{s_k}), \end{aligned}$$

and the family $x_{s_j s_k}(X)$, $X \in L_p(\mathcal{F}_{s_k})$ with $s_j, s_k \in \mathcal{T} : s_j \leq s_k$ is time-consistent. Moreover for every $t \in [s_{k-1}, s_k]$ and $X \in L_t^+ \subseteq L_p(\mathcal{F}_t)$, (2.8) and the \mathcal{F}_t -homogeneity give

$$\begin{aligned} x_{s_{k-1}t}(X) &= x_{s_{k-1}t}(X x_{ts_k}(1)) = x_{s_{k-1}t}(x_{ts_k}(X)) = x_{s_{k-1}s_k}(X) \\ &= E\left[X \frac{f_{s_{k-1}s_k}}{E[f_{s_{k-1}s_k}|\mathcal{F}_{s_{k-1}}]} \middle| \mathcal{F}_{s_{k-1}}\right] = E\left[X \frac{f}{E[f|\mathcal{F}_{s_{k-1}}]} \middle| \mathcal{F}_{s_{k-1}}\right]. \end{aligned}$$

Naturally for $t \in [s_k, s_{k+1}]$ and $X \in L_t^+ \subseteq L_p(\mathcal{F}_{s_{k+1}})$, we have

$$\begin{aligned} x_{s_{k-1}t}(X) &= x_{s_{k-1}s_k}(x_{s_k t}(X x_{ts_{k+1}}(1))) = x_{s_{k-1}s_k}(x_{s_k s_{k+1}}(X)) \\ &= E\left[X \frac{f_{s_{k-1}s_k}}{E[f_{s_{k-1}s_k}|\mathcal{F}_{s_{k-1}}]} \frac{f_{s_k s_{k+1}}}{E[f_{s_k s_{k+1}}|\mathcal{F}_{s_k}]} \middle| \mathcal{F}_{s_{k-1}}\right] \\ &= E\left[X \frac{f}{E[f|\mathcal{F}_{s_{k-1}}]} \middle| \mathcal{F}_{s_{k-1}}\right]. \end{aligned}$$

Hence, iterating the argument we can conclude that the probability measure

$$P^0(A) = \int_A f(\omega) P(d\omega), \quad A \in \mathcal{F}_T, \quad (4.8)$$

allows the representation

$$x_{st}(X) = E\left[X \frac{f}{E[f|\mathcal{F}_s]} \middle| \mathcal{F}_s\right], \quad X \in L_t^+,$$

for every $s \in \mathcal{T}$ and $t \in [s, T]$. Moreover, from Theorem 3.3, we have

$$m = \prod_{k=1}^K m_{s_{k-1}s_k} \leq \prod_{k=1}^K \frac{f_{s_{k-1}s_k}}{E[f_{s_{k-1}s_k}|\mathcal{F}_{s_{k-1}}]} \leq \prod_{k=1}^K M_{s_{k-1}s_k} = M.$$

Thus $\mathbb{P}^{\mathcal{T}}$ is non-empty and A holds.

To prove B we consider, for each n , the set

$$\begin{aligned} \mathbb{D}^{(\mathcal{T}_n)} &:= \left\{ f \in L_q^+(\mathcal{F}) \mid E[f|\mathcal{F}_0] = 1, m \leq f \leq M; \right. \\ &\quad \left. \forall s \in \mathcal{T}_n, t \in [s, T], x_{st}(X) = E\left[X \frac{f}{E[f|\mathcal{F}_s]} \middle| \mathcal{F}_s\right] \forall X \in L_t^+ \right\} \end{aligned}$$

of the densities corresponding to $\mathbb{P}^{(\mathcal{T}_n)}$. We show that $\mathbb{D}^{(\mathcal{T}_n)}$ is compact with respect to the weak* topology. Then applying the finite intersection property, we can conclude that $\bigcap_{n=1}^{\infty} \mathbb{D}^{(\mathcal{T}_n)} \neq \emptyset$ and thus $\bigcap_{n=1}^{\infty} \mathbb{P}^{(\mathcal{T}_n)} \neq \emptyset$. Recall that we are dealing with separable L_p -spaces $1 \leq p < \infty$ where the concepts of weak* closed and weak* sequentially closed are equivalent. Then it is enough to show that $\mathbb{D}^{(\mathcal{T}_n)}$ is weak* sequentially closed and bounded in norm, see e.g. [16, Chapter 12, theorem 3].

Let $\tau = \tau_n$ be a partition of type (4.6). $\mathbb{D}^{(\mathcal{T})}$ is bounded by definition. Let us consider a sequence $\{f_j\}$ of elements in $\mathbb{D}^{(\mathcal{T})}$ converging to $f \in L_q(\mathcal{F})$ in the weak* sense, i.e. $E[Xf_j] \rightarrow E[Xf]$ for all $X \in L_p(\mathcal{F})$. We prove that $f \in \mathbb{D}^{(\mathcal{T})}$.

Clearly

$$E[f|\mathcal{F}_0] = 1. \quad (4.9)$$

Further, for any n , $s \in \mathcal{T}$, $t \in [s, T]$ and $X \in L_t^+$,

$$E[f_j X | \mathcal{F}_s] = x_{st}(X) E[f_j | \mathcal{F}_s],$$

i.e. $E[\chi_A X f_j] = E[\chi_A x_{st}(X) f_j]$, for any $A \in \mathcal{F}_s$. Letting $j \rightarrow \infty$, we have

$$E[\chi_A X f] = E[\chi_A x_{st}(X) f]$$

and

$$E[\chi_A E[X f | \mathcal{F}_s]] = E[\chi_A x_{st}(X) E[f | \mathcal{F}_s]]$$

for any $A \in \mathcal{F}_s$. Namely,

$$E[X f | \mathcal{F}_s] = x_{st}(X) E[f | \mathcal{F}_s].$$

Hence we have

$$E\left[\frac{fX}{E[f | \mathcal{F}_s]} \middle| \mathcal{F}_s\right] = x_{st}(X) = E\left[\frac{f_j X}{E[f_j | \mathcal{F}_s]} \middle| \mathcal{F}_s\right], \quad X \in L_t^+. \quad (4.10)$$

Now, we fix $\delta > 0$ and let

$$A := \left\{ \omega; f \leq m - \delta \right\}.$$

Then,

$$E[f \chi_A | \mathcal{F}_0] \leq E[(m - \delta) \chi_A | \mathcal{F}_0] = E[m \chi_A | \mathcal{F}_0] - \delta E[\chi_A | \mathcal{F}_0],$$

while

$$E[f_j \chi_A | \mathcal{F}_0] \geq E[m \chi_A | \mathcal{F}_0].$$

For (4.10) to hold we must have $P(A) = 0$. Thus $f > m - \delta$ P -a.s. Letting $\delta \rightarrow 0$, we obtain

$$f \geq m. \quad (4.11a)$$

By replacing " $\leq m - \delta$ " with " $\geq M + \delta$ " in the definition of A and proceeding similarly we get that

$$f \leq M. \quad (4.11b)$$

Then (4.9)-(4.11) ensure that $f \in \mathbb{D}^{(T)}$, which is then closed with respect to the weak* convergence, and thus weak* (sequentially) compact. This concludes the proof of B.

Assume that $P^0 \in \bigcap_{n=1}^{\infty} \mathbb{P}^{(\mathcal{T}_n)}$. As the partitions form a dense subset of $[0, T]$, then for any $s \in [0, T]$ there is a sequence $s_n \in \mathcal{T}_n$, $n = 1, 2, \dots$ such that $s_n \downarrow s$ as $n \rightarrow \infty$. By the right-continuity (2.7) of the price operators and the right-continuity of the filtration we have

$$x_{st}(X) = \lim_{n \rightarrow \infty} x_{s_n t}(X) = \lim_{n \rightarrow \infty} E^0[X | \mathcal{F}_{s_n}] = E^0[X | \mathcal{F}_s], \quad X \in L_t^+,$$

for any $s, t \in [0, T]$, $s \leq t$. Thus $P^0 \in \mathbb{P}$ and C holds. \square

5 Examples

Example 5.1. In a single period market model with $T > 0$ as time horizon we consider the insurance claim

$$H = \int_{z_0}^{\infty} (z - z_0) N((0, T], dz),$$

which could be interpreted as a contract that covers all losses (fires, car accidents etc.) exceeding the (deductible) amount $z_0 > 0$ in a specific time period $[0, T]$. The number of losses of a magnitude $(z, z + dz)$, is modelled by the Poisson random variable $N((0, T], dz)$ with $E[N((0, T], dz)] = T\nu(dz)$. This is the value of the Poisson random measure N on the set $(0, T] \times (z, z + dz]$ with the σ -finite Borel measure $\nu(dz)$, $z > 0$, representing the jump behaviour. We assume that $\mathcal{F} = \mathcal{F}_T$ is generated by the random variables $N((0, T], (a, b])$, $a \leq b$, and \mathcal{F}_0 is trivial. Applying the expected value principle (see e.g. [23]) with loading factor δ , the price of this contract would be

$$x(H) = x_{0T}(H) = (1 + \delta)E[H] = (1 + \delta)T \int_{z_0}^{\infty} (z - z_0)\nu(dz). \quad (5.1)$$

Normally one would have $\delta > 0$, but to keep our approach more general, we will only assume $\delta > -1$ to have strictly positive prices. The expected value principle only makes sense if we assume that the intensities are "sufficiently nice" for H to belong to L_1 . Whether higher order moments exist depends on the jump-size intensity ν . We assume that $H \in L_p(\mathcal{F})$ for some $p \geq 1$.

We assume that any number or fraction $\alpha \geq 0$ of the claim is available at a proportional price. Alternatively, the investor can buy some riskless security with no interest. The admissible claims thus belong to the convex cone $L^+ = \{\alpha H + \beta : \alpha, \beta \geq 0\} \subseteq L_p^+(\mathcal{F})$ for some $p \geq 1$. We consider the price operator x defined on L^+ by

$$x(X) = \beta + \alpha(1 + \delta)T \int_{z_0}^{\infty} (z - z_0)\nu(dz), \quad X = \alpha H + \beta. \quad (5.2)$$

Clearly x is strictly monotone, additive and scale invariant (i.e. \mathcal{F}_0 -homogeneous). Any martingale measure $P^0 \sim P$ with $f = \frac{dP^0}{dP}$ is characterized by

$$x(X) = E^0[X] = E[Xf], \quad X \in L^+.$$

According to Theorem 4.1, given the random variables $m, M \in L_q^+(\mathcal{F})$ with $0 < m \leq M$ P -a.s., if the sandwich condition

$$E[mY''] + x(X'') \leq x(X') + E[MY'], \quad (5.3)$$

holds for all random variables $Y'', Y' \in L_p^+(\mathcal{F})$ and $X', X'' \in L^+$ such that

$$Y'' + X'' \leq X' + Y', \quad (5.4)$$

then x can be extended to the whole $L_p(\mathcal{F})$ and there exists $P^0 \sim P$ such that the extension has the representation

$$x(X) = E^0[X] = E[Xf], \quad X \in L_p(\mathcal{F}), \quad (5.5)$$

and

$$m \leq f \leq M \quad (5.6)$$

The converse is also true. Naturally the extension (5.5) is then lying within the bounds

$$E[mX] \leq x(X) \leq E[MX], \quad X \in L_p(\mathcal{F}). \quad (5.7)$$

Let us consider the case $m = M = 1$. Then the only density satisfying (5.6) is $f = 1$. Let $P^0 := fdP$. If P^0 was a martingale measure, then from (5.5) and (5.2) we would have

$$E[H] = x(H) = (1 + \delta)E[H]. \quad (5.8)$$

However, if $\delta > 0$, then (5.8) is absurd and P^0 is not an equivalent martingale measure for the prices (5.2). (Fact that was easy to see directly!) If $\delta = 0$, then (5.8) is trivially verified, as well as the observation that P^0 is in this case an equivalent martingale measure for the prices (5.2).

The upper and lower bounds are usually thought of as exogenously given. The sandwich condition fully characterizes when the set \mathbb{P} (4.4) of equivalent martingale measures with density satisfying (5.6) is non-empty. Hereafter we discuss some "reasonable" non-trivial lower and upper bounds leading to a non-empty set \mathbb{P} . By decomposing the claims, we can rewrite the sandwich condition (5.3) as

$$0 \leq E[MY' - mY''] + (1 + \delta)(\alpha' - \alpha'')E[H] + \beta' - \beta'' \quad (5.9)$$

and (5.4)

$$Y'' \leq Y' + (\alpha' - \alpha'')H + \beta' - \beta''. \quad (5.10)$$

Consider first the "non-actuarial" case $\delta < 0$. Note that (5.9) becomes

$$E[mH] \leq (1 + \delta)E[H]$$

when we consider $Y'' = Y' + (\alpha' - \alpha'')H + \beta' - \beta''$ and then set $Y' = 0, \beta' = \beta''$. The inequality above holds if, for example,

$$m = (1 + \delta)^{N((0,T], I_0)}, \quad I_0 := (z_0, \infty).$$

In fact we have

$$E[mH] = (1 + \delta)e^{\delta T\nu(I_0)}T \int_{z_0}^{\infty} (z - z_0)\nu(dz) = (1 + \delta)e^{\delta T\nu(I_0)}E[H],$$

where to performe the computation we have applied the fact that Poisson random measures have independent values, in particular $N(0, T], (a, b]$ is independent of $N(0, T], (c, d]$, whenever $(a, b] \cap (c, d] = \emptyset$.

We must also have $M \geq m$ and $P(M > 1) > 0$ (see (5.7) with $X = 1$). We may choose

$$M = e^{-\delta T\nu(I_0)}.$$

By proceeding similarly in the case where $\delta > 0$, we get that

$$m = e^{-\delta T\nu(I_0)}, \quad M = (1 + \delta)^{N((0, T], I_0)} \quad (5.11)$$

are suitable bounds. For these bounds the set \mathbb{P} is non-empty. For example, in the case $\delta > 0$ with (5.11), consider the martingale measures $P_1^0 \sim P$, i.e. $E_1^0[X] = E[Xf_1] = x(X)$, $X \in L^+$, of structure preserving nature (see e.g. [5]), i.e. the random variable $N((0, T], dz)$ has a Poisson distribution with $E_1^0[N((0, T], dz)] = T\mu_1(dz)$ where

$$\mu_1(dz) = (1 + \delta)\nu(dz).$$

In this setting the densities $f_1 = \frac{dP_1^0}{dP}$ is given by

$$f_1 = (1 + \delta)^{N((0, T], I_0)} e^{-\delta T\nu(I_0)}, \quad I_0 = (z_0, \infty).$$

For the given bounds (5.11), P_1^0 satisfies (5.6) and $P_1^0 \in \mathbb{P}$.

We remark that not all equivalent martingale measures are contained in \mathbb{P} . For example, the structure preserving martingale measure $P_2^0 \sim P$ with density $f_2 = \frac{dP_2^0}{dP}$:

$$f_2 = (1 + \gamma)^{N((0, T], I^*)} e^{-\gamma T\nu(I^*)}, \quad I^* = (z^*, \infty).$$

i.e. the martingale measure such that $E_2^0[N((0, T], dz)] = T\mu_2(dz)$ with

$$\mu_2(dz) = \begin{cases} \nu(dz), & z \leq z^* \\ (1 + \gamma)\nu(dz), & z > z^* \end{cases}, \quad \gamma = \delta \frac{\int_{z_0}^{\infty} (z - z_0)\nu(dz)}{\int_{z^*}^{\infty} (z - z_0)\nu(dz)} \quad (z^* > z_0),$$

belongs to \mathbb{P} only depending on the choice of the involved parameters.

The choice of equivalent martingale measure is of more than academic interest: it is crucial that the applied measures do not underestimate the probabilities of the events covered by the contract, e.g. if one wants to assign prices to contracts with different deductibles, or to calculate prices of excess of loss reinsurance contracts covering catastrophic losses.

Example 5.2. At any time t , the claim covering losses (exceeding $z_0 > 0$) up to time t :

$$H_t = \int_{z_0}^{\infty} (z - z_0) N((0, t], dz),$$

is available at time s ($s \leq t$) at a price that equals the losses already incurred plus the uncertain part of the claim priced according to the expected value principle:

$$x_{st}(H_t) = \int_{z_0}^{\infty} (z - z_0) N((0, s], dz) + (1 + \delta)(t - s) \int_{z_0}^{\infty} (z - z_0) \nu(dz). \quad (5.12)$$

Here N is a Poisson random measure and ν is the measure representing the jump behaviour. In this market, the claims available are $L_t^+ := \{\alpha H_t + \beta : \alpha, \beta \geq 0\}$, $t \in [0, T]$. We assume that the filtration \mathcal{F}_t , $t \in [0, T]$, is generated by the random values of $N(ds, dz)$, $0 \leq s \leq t$ and $z > 0$ and \mathcal{F}_0 is trivial. We assume that $L_t^+ \subseteq L_p^+(\mathcal{F}_t)$ for some $p \geq 1$ according to the choice of ν . For any $X \in L_t^+$, the prices are given by

$$x_{st}(X) = \alpha x_{st}(H_t) + \beta.$$

As in the previous example, for any $s, t \in [0, T]$, $s \leq t$, we find some m_{st} , M_{st} such that

$$0 \leq E[(M_{st}Y' - m_{st}Y'')|\mathcal{F}_t] + (\alpha' - \alpha'')x_{st}(H_t) + \beta' - \beta'', \quad 0 \leq t \leq T,$$

for all $Y', Y'' \in L_p^+(\mathcal{F}_t)$ and $X', X'' \in L_t^+$ such that $Y'' \leq Y' + (\alpha' - \alpha'')H_t + \beta' - \beta''$. By proceeding as in Example 5.1, we see that

$$m_{st} = e^{-\delta(t-s)\nu(I_0)}, \quad M_{st} = (1 + \delta)^{N((s, t], I_0)}, \quad (5.13)$$

are suitable bounds in the case $\delta > 0$. By Theorem 4.1 there exists some martingale measure $P^0 \sim P$ whose density $f = \frac{dP^0}{dP}$ satisfies

$$m = m_{0T} = e^{-\delta T\nu(I_0)} \leq f \leq (1 + \delta)^{N((0, T], I_0)} = M_{0T} = M.$$

For example, the structure preserving martingale measure such that all $N((s, t], dz)$ are Poisson distributed with $E^0[N((s, t], dz)] = (t - s)\mu(dz)$ where $\mu(dz) = (1 + \delta)\nu(dz)$ has density

$$f = (1 + \delta)^{N((0, T], I_0)} e^{-\delta T\nu(I_0)}, \quad I_0 = (z_0, \infty),$$

lying within the given bounds and belonging to \mathbb{P} .

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